GOOD COVERINGS

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In the cohomology theory of differentiable manifolds, e.g. in the proof of the De Rham-theorem and the treatment of Poincaré-duality, it is used that every open covering of a paracompact differentiable manifold has a refinement $\mathcal{U}=\{U_i\}_{i\in I}$ with the property that each non-empty finite intersection $U_{i_0}\cap\dots\cap U_{i_k}$ is diffeomorphic with R^n. A covering with this property is called "good". The existence of such good refinements is mostly proved by referring to some theorems in differential geometry, which are out of the scope of cohomology theory. The aim of this paper is to show how to obtain good coverings in an elementary way, without using differential geometry. To start I prove the following theorem:

THEOREM 1: Let $x_0\in E\subset R^n$, E open, $V\subset R^n$ and $f:E\to V$ be a homeomorfism with both f and f^{-1} C^2-functions. Then there exists a $\delta^*>0$ such that for every $\delta>0$ with $\delta<\delta^*$, the open ball $B(x_0,\delta)=\{x\in R^n, |x-x_0|<\delta\}$ is entirely contained in E and $f(B(x_0,\delta))$ is a convex subset of V.

PROOF: Taylor's formula tells us that if $V^*\subset V\subset R^n$, with V open, V^* compact and convex, and $f:V\to R^n$ is a C^m-function, then for each $x,h\in R^n$ with both x and $x+h\ \in V^*$, some $\theta=(\theta_1,\dots,\theta_n)\ \in R^n$ with $0<\theta_i<1$ exists such that

(1) $$f(x+h) = f(x) + \frac{1}{1!}D^1f(x) + \dots + \frac{1}{(n-1)!}D^{n-1}f(x) + R_m(x,h) \quad \text{with}$$

$$D^kf(x) = \sum_{i_1\dots i_k=1}^{n} h_{i_1}\dots h_{i_k}\cdot\frac{\partial^k f}{\partial x_{i_1}\dots\partial x_{i_k}}(x) \text{ and } R_m(x,h) = \frac{1}{m!}D^mf(x+\theta h).$$

Furthermore, since each continuous function has a maximum and a minimum on V^*, there is a $M_m\in R^+$ such that

$$\left|\frac{\partial^m f_i}{\partial x_{i_1}\dots\partial x_{i_m}}(x)\right|<M_m \text{ for each } x\in V^* \text{ and each } 1<i_1,\dots,i_m,i<n.$$

This implies that for each component $R_{m,i}(x,h)\in R$ of $R_m(x,h)\in R^n$ we have

$$\left|R_{m,i}(x,h)\right| < \frac{1}{m!}\sum_{i_1\dots i_m=1}^{n}\left|h_{i_1}\right|\dots\left|h_{i_m}\right|\cdot\left|\frac{\partial^m f_i}{\partial x_{i_1}\dots\partial x_{i_m}}(x+\theta h)\right| < \frac{n^m}{m!}|h|^m M_m$$

so that, for all $x,h\in R^n$ for which both x and $x+h\ \in V^*$:

$$(2)\quad |R_m(x,h)| = \sqrt{\sum_{i=1}^{n} |R_{m,i}|^2} < \frac{\sqrt{n}}{m!} n^m M_m |h|^m = K_m |h|^m, \text{ with } K_m = \frac{\sqrt{n}}{m!} n^m M_m.$$

Now let x_0, E, V and f be as in the theorem. Without limiting the generality of the theorem, I may assume that $x_0=0$. Choose $V^* \subset V$ compact and convex, such that $f(0) \in \text{Interior}(V^*)$, and $\varepsilon>0$, such that $B(0,\varepsilon) \subset f^{-1}(V^*) \subset E$. From (1) and (2) I conclude the following. Firstly, there exists some $K_1 \in R^+$, such that for each z and $z+k \in V^*$

$$(3)\quad |f^{-1}(z+k)| < |f^{-1}(z)| + K_1|k|$$

and secondly, there exists some $K_2 \in R^+$ such that for each $u=x+h \in B(0,\varepsilon)$ and $v=x-h \in B(0,\varepsilon)$

$$f(x+h) = f(x) + \sum_{j=1}^{n} h_j \frac{\partial f}{\partial x_j}(x) + R_2(x,h) \quad \text{with } |R_2(x,h)| < K_2|h|^2 \text{ and}$$

$$f(x-h) = f(x) - \sum_{j=1}^{n} h_j \frac{\partial f}{\partial x_j}(x) + R_2(x,-h) \text{ with } |R_2(x,-h)| < K_2|h|^2.$$

Hence $\frac{1}{2}f(x+h) + \frac{1}{2}f(x-h) = f(x) + (\frac{1}{2}R_2(x,h) + \frac{1}{2}R_2(x,-h))$ with

$|\frac{1}{2}R_2(x,h) + \frac{1}{2}R_2(x,-h)| < K_2|h|^2$ or, in other words:

$$(4)\quad \tfrac{1}{2}f(u) + \tfrac{1}{2}f(v) = f(\tfrac{u+v}{2}) + R^*(u,v) \text{ with } |R^*(u,v)| < K_2|\tfrac{u-v}{2}|^2.$$

Note that, when u and v are chosen in $B(0,\varepsilon)$, then both $f(\frac{u+v}{2})$ and $\frac{1}{2}f(u)+\frac{1}{2}f(v)$ lie in V^*, so that (3) and (4) can be combined to:

$$(5)\quad |f^{-1}(\tfrac{1}{2}f(u) + \tfrac{1}{2}f(v))| \underset{(4)}{=} |f^{-1}(f(\tfrac{u+v}{2}) + R^*(u,v))| \underset{(3)}{<}$$

$$|f^{-1}(f(\tfrac{u+v}{2}))| + K_1|R^*(u,v)| \underset{(4)}{<} |\tfrac{u+v}{2}| + K_1K_2|\tfrac{u-v}{2}|^2.$$

Now let δ^* be the minimum of ε and $\frac{1}{2K_1K_2}$. Then it is sufficiënt to prove that for each $0<\delta<\delta^*$: $f(B(0,\delta))$ is convex. Note that for an open set S to be convex it suffices that $x,y \in S$ implies $\frac{x+y}{2} \in S$; this because $x,y \in S$ implies that $B(x,\delta) \subset S$ and $B(x,\delta) \subset S$ for some $\delta \in R^+$, so that also $B(\frac{x+y}{2},\delta) \subset S$, and by repeating this argument a finite times it turnes out that the entire linesegment $\{\lambda x+(1-\lambda)y,\ \lambda \in [0,1]\}$ is contained in S.

So let $0<\delta<\delta^*$ and $x,y \in f(B(0,\delta))$; it must be proved that also $\frac{x+y}{2} \in f(B(0,\delta))$.

Set $u=f^{-1}(x)$ and $v=f^{-1}(y)$, then $u,v\in B(0,\delta)$, or $|u|<\delta$ and $|v|<\delta$. I have to prove that $\frac{1}{2}f(u)+\frac{1}{2}f(v)\in f(B(0,\delta))$, which means $f^{-1}(\frac{1}{2}f(u)+\frac{1}{2}f(v))\in B(0,\delta)$ or $|f^{-1}(\frac{1}{2}f(u)+\frac{1}{2}f(v))|<\delta$. From (5) it follows that
$|f^{-1}(\frac{1}{2}f(u)+\frac{1}{2}f(v))| < |\frac{u+v}{2}| + \frac{1}{2\delta}|\frac{u-v}{2}|^2$, so it suffices to prove that
$|\frac{u+v}{2}| < \delta - \frac{1}{2\delta}|\frac{u-v}{2}|^2$. Since $|u|<\delta$ and $|v|<\delta$ the righthand side of the last inequality is positive and therefore the inequality is equivalent with:
$|\frac{u+v}{2}|^2 < (\delta - \frac{1}{2\delta}|\frac{u-v}{2}|^2)^2)$.

This can be proved by using the cosinus-rule in the following way:
$|\frac{v+u}{2}|^2 + |\frac{v-u}{2}|^2 - 2|\frac{v+u}{2}|.|\frac{v-u}{2}|\cos(\frac{v+u}{2},\frac{v-u}{2}) = |v|^2 < \delta^2$;
$|\frac{u+v}{2}|^2 + |\frac{u-v}{2}|^2 - 2|\frac{u+v}{2}|.|\frac{u-v}{2}|\cos(\frac{u+v}{2},\frac{u-v}{2}) = |u|^2 < \delta^2$.
Since either $\cos(\frac{u+v}{2},\frac{u-v}{2})<0$ or $\cos(\frac{v+u}{2},\frac{v-u}{2})<0$ it follows that
$|\frac{u+v}{2}|^2 + |\frac{u-v}{2}|^2 < \delta^2$, and this implies that
$|\frac{u+v}{2}|^2 < \delta^2 - |\frac{u-v}{2}|^2 < \delta^2 - |\frac{u-v}{2}|^2 + (\frac{1}{2\delta}|\frac{u-v}{2}|^2)^2 = (\delta - \frac{1}{2\delta}|\frac{u-v}{2}|^2)^2$, which had to be proved.

THEOREM 2: Any nonempty open convex subset of R^n is diffeomorphic with R^n itself.

PROOF: It can even be proved that for any $\psi\in C^0(S^{n-1},R)$ with $\psi>0$ the set $B = \{rw,\ w\in S^{n-1},\ r\in R,\ 0<r<\psi(w)\}$ is diffeomorphic to $E^n=\{x\in R^n, |x|<1\}$, and therefore to R^n; and since any nonempty open convex subset of R^n is diffeomorphic to such a set B the theorem follows.

Let $\psi\in C^0(S^{n-1},R)$, $\psi>0$ be given and let $\psi_*\in R^+$ be the minimum of ψ on S^{n-1}. Now, for $k=2,3,4,\ldots$ choose $\psi_k\in C^\infty(S^{n-1},R)$ such that $(1-2^{-2k})\psi < \psi_k < (1-2^{-2k-1})\psi$, and let $\psi_1\in C^\infty(S^{n-1},R)$ be given by $\psi_1(w) = \frac{1}{4}\psi_*$.
Furthermore, for $k\in N^+$ define $h_k\in C^\infty(S^{n-1},R)$ by $h_k(w) = \psi_{k+1}(w) - \psi_k(w) - 2^{-2(k+1)}\psi_*$ and $g_k\in C^\infty((1-2^{-2k},1-2^{-2(k+1)}),R)$ by

$$g_k(r) = \frac{e^{1/(r-1+2^{-2k})(r-1+2^{-2(k+1)})}}{\int_{y=1-2^{-2k}}^{1-2^{-2(k+1)}} e^{1/(y-1+2^{-2k})(y-1+2^{-2(k+1)})}dy}.$$

Note, that for $k\in N^+$: $\psi_k < (1-2^{-2k-1})\psi < (1-2^{-2k-2})\psi < \psi_{k+1}$, hence

$\psi_{k+1}-\psi_k > (1-2^{-2k-2})\psi - (1-2^{-2k-1})\psi = 2^{-2k-2}\psi > 2^{-2(k+1)}\psi_*$, and $h_k>0$.

Note also, that for $k\in N^+$ and $i=0,1,2,\ldots$

I $\quad \lim_{r\downarrow 1-2^{-2k}} \int_{1-2^{-2k}}^{r} g_k(y)dy = 0$ and $\lim_{r\uparrow 1-2^{-2k}} \int_{1-2^{-2(k-1)}}^{r} g_{k-1}(y)dy = 1$

II $\quad \lim_{r\downarrow 1-2^{-2k}} \frac{d^i g_k}{dr^i}(r) = \lim_{r\uparrow 1-2^{-2k}} \frac{d^i g_{k-1}}{dr^i}(r) = 0$

III $\quad \lim_{t\downarrow 0} \frac{1}{t} \int_{1-2^{-2k}}^{1-2^{-2k}+t} g_k(y)dy = \lim_{t\uparrow 0} \frac{1}{t} \int_{1-2^{-2k}}^{1-2^{-2k}+t} g_{k-1}(y)dy = 0$

IV $\quad \lim_{t\downarrow 0} \frac{1}{t} \frac{d^i g_k}{dt^i}(1-2^{-2k}+t) = \lim_{t\uparrow 0} \frac{1}{t} \frac{d^i g_{k-1}}{dt^i}(1-2^{-2k}+t) = 0$.

From here on I use polar coördinates (r,w) with $r\in[0,\rightarrow)$ and $w=(w_1,\ldots,w_{n-1})\in S^{n-1}$.

Define $f:E\rightarrow R^n$ by:

$$f_w(r,w) = w$$

$$f_r(r,w) = \begin{cases} r\cdot\frac{\psi_*}{3} & \text{if } r<\frac{3}{4} \\ \psi_k(w) & \text{if } r=1-2^{-2k} \text{ with } k\in N^+ \\ \psi_k(w) + \int_{y=1-2^{-2k}}^{r} \frac{\psi_*}{3} + h_k(w)g_k(y)dy & \text{if } 1-2^{-2k}<r<1-2^{-2(k+1)} \text{ with } k\in N^+. \end{cases}$$

I shall prove that f is a diffeomorphism from E to B.

1. $\quad$ <u>f_r is continuous.</u>

In order to show that f_r is continuous in $(r_0,w_0)\in E$, one has to check that $\lim_{\substack{r\rightarrow r_0\\ w\rightarrow w_0}} f(r,w) = f_r(r_0,w_0)$. This is trivial for $1-2^{-2k}<r_0<1-2^{-2(k+1)}$ $(k\in N^+)$, and for $r_0<\frac{3}{4}$ and since $\lim_{w\rightarrow w_0} f_r(r,w) = f_r(r,w_0)$ I only have to check that $\lim_{r\rightarrow r_0} f_r(r,w_0) = f_r(r_0,w_0)$ for $r_0=1-2^{-2k}$ $(k\in N^+)$, or that $\lim_{r\downarrow 1-2^{-2k}} f_r(r,w) = \lim_{r\uparrow 1-2^{-2k}} f_r(r,w) = \psi_k(w)$. Using "I" one finds:

$\lim_{r\downarrow 1-2^{-2k}}[\psi_k(w) + \frac{\psi_*}{3}\{r-(1-2^{-2k})\} + h_k(w) \int_{y=1-2^{-2k}}^{r} g_k(y)dy] = \psi_k(w)$ and

for $k=1$: $\lim_{r\uparrow 1-2^{-2k}} r\cdot\frac{\psi_*}{3} = \frac{3}{4}\cdot\frac{\psi_*}{3} = \frac{1}{4}\psi_* = \psi_k(w)$,

for $k\geq 2$: $\lim_{r\uparrow 1-2^{-2k}}[\psi_{k-1}(w) + \frac{\psi_*}{3}\{r-(1-2^{-2(k-1)})\} + h_{k-1}(w) \int_{1-2^{-2(k-1)}}^{r} g_{k-1}(y)dy]=$

$$\psi_{k-1}(w) + \frac{\psi_*}{3}(2^{-2k+2} - 2^{-2k}) + h_{k-1}(w) =$$

$$\psi_{k-1}(w) + \frac{\psi_*}{3}.3.2^{-2k} + \psi_k(w) - \psi_{k-1}(w) - 2^{-2k}\psi_* = \psi_k(w),$$

so f_r is continuous.

2. <u>$B \subset f(E)$.</u>

Let $(r_0,w_0) \in B$ be given, so $0 < r_0 < \psi(w_0)$. Choose $k \in N^+$ in such a way that $r_0 < (1-2^{-2k})\psi(w_0) < \psi_k(w_0)$. Now $f_r|_{w=w_0}$ is a continuous function on a connected set, taking on the values 0 (in $r=0$) and $\psi_k(w_0)$ (in $r=1-2^{-2k}$), and therefore also taking on the value r_0, say in r_1. Now $(r_1,w_0) \in E$ and $f(r_1,w_0) = (r_0,w_0)$.

3. <u>f_r is differentiable with respect to r and $\frac{\partial f_r}{\partial r} > 0$ on E.</u>

This is again trivial for $r < \frac{3}{4}$ and for $1-2^{-2k} < r < 1-2^{-2(k+1)}$ with $k \in N^+$ so take $r=1-2^{-2k}$ $(k \in N^+)$.

$$\lim_{t\downarrow 0} \frac{f_r(r+t,w) - f_r(r,w)}{t} = \lim_{t\downarrow 0} \frac{1}{t} \int_{y=1-2^{-2k}}^{1-2^{-2k}+t} \frac{\psi_*}{3} + h_k(w)g_k(y)dy =$$

$$\lim_{t\downarrow 0} \frac{1}{t}(\frac{\psi_*}{3}t) + h_k(w) \lim_{t\downarrow 0} \frac{1}{t} \int_{1-2^{-2k}}^{1-2^{-2k}+t} g_k(y)dy = \frac{\psi_*}{3} \text{ (using III).}$$

For $k=1$: $$\lim_{t\uparrow 0} \frac{f_r(\frac{3}{4}+t,w) - f_r(\frac{3}{4},w)}{t} = \lim_{t\uparrow 0} \frac{\frac{\psi_*}{3}(\frac{3}{4}+t) - \frac{\psi_*}{4}}{t} = \frac{\psi_*}{3}.$$

For $k \geq 2$: $$\lim_{t\uparrow 0} \frac{f_r(r+t,w) - f_r(r,w)}{t} =$$

$$\lim_{t\uparrow 0} \frac{1}{t}[\psi_{k-1}(w) + \int_{y=1-2^{-2k+2}}^{1-2^{-2k}+t} \frac{\psi_*}{3} + h_{k-1}(w)g_{k-1}(y)dy - \psi_k(w)] =$$

$$\lim_{t\uparrow 0} \frac{1}{t}[\psi_{k-1}(w) + \int_{1-2^{-2k+2}}^{1-2^{-2k}} \frac{\psi_*}{3} + h_{k-1}(w)g_{k-1}(y)dy - \psi_k(w)] +$$

$$\lim_{t\uparrow 0} \frac{1}{t}[\int_{1-2^{-2k}}^{1-2^{-2k}+t} \frac{\psi_*}{3} + h_{k-1}(w)g_{k-1}(y)dy] = \text{(using III)}$$

$$\lim_{t\uparrow 0} \frac{1}{t}[\psi_{k-1}(w)+\frac{\psi_*}{3}.3.2^{-2k}+h_{k-1}(w)-\psi_k(w)] + \lim_{t\uparrow 0} \frac{1}{t}(\frac{\psi_*}{3}t) + h_{k-1}(w).0 = \frac{\psi_*}{3}.$$

So f_r is differentiable with respect to r and $\frac{\partial f_r}{\partial r}(1-2^{-2k}) = \frac{\psi_*}{3}$ $(k \in N^+)$.

$$\frac{\partial f_r}{\partial r}(r,w) = \begin{cases} \frac{\psi_*}{3} & \text{if } r<\frac{3}{4} \text{ and if } r=1-2^{-2k} \ (k\in N^+) \\ \frac{\psi_*}{3} + h_k(w)g_k(r) & \text{if } 1-2^{-2k}<r<1-2^{-2(k+1)} \ (k\in N^+); \end{cases}$$

and since $\psi_*>0$, $h_k \geq 0$ and $g_k \geq 0$ it follows that $\frac{\partial f_r}{\partial r}>0$ on E.

4. $f(E) \subset B$.

Let $(r_0,w_0) \subset E$ be given, so $r_0<1$. Choose $k\in N^+$ in such a way that $r_0 < 1-2^{-2k}$. Now $f_r|_{w=w_0}$ is an increasing function (because $\frac{\partial f_r}{\partial r}>0$) on a connected set, taking on the values 0 (in r=0) and $\psi_k(w_0)$ (in $r=1-2^{-2k}$), so it follows that $0 < f_r(r_0,w_0) < \psi_k(w_0) < \psi(w_0)$ and $f(r_0,w_0)\in B$.

5. f is injective.

Suppose $f(r_0,w_0) = f(r_1,w_1) = (r_2,w_2)$ then $w_0 = w_1 = w_2$ and $f_r|_{w=w_2}: [0,1)\to R$ is an increasing function, which takes on the value r_2 only once.

6. f is C^∞.

For $x\in R^n$, $|x|<\frac{3}{4}$, f is a lineair function given by $f(x)=x.\frac{\psi_*}{3}$. Therefore f certainly is C^∞ in r=0. f is also C^∞ on the rest of E^n if for n=0,1,... all n^{th}-order partial derivatives exist and are continuous on $E^n\setminus\{0\}$. For f_w this is trivial, for f_r the partial derivatives turn out to be:

$$\frac{\partial^j f_r}{\partial w_{i_1}\cdots\partial w_{i_j}}(r,w) = \begin{cases} 0 & \text{if } r<\frac{3}{4} \\ \frac{\partial^j \psi_k}{\partial w_{i_1}\cdots\partial w_{i_j}}(w) & \text{if } r=1-2^{-2k} \\ \frac{\partial^j \psi_k}{\partial w_{i_1}\cdots\partial w_{i_j}}(w) + \frac{\partial^j h_k}{\partial w_{i_1}\cdots\partial w_{i_j}}(w) \int_{1-2^{-2k}}^{r} g_k(y)\,dy & \text{if } 1-2^{-2k}<r<1-2^{-2(k+1)} \end{cases} \quad (j\geq 1)$$

$$\frac{\partial f_r}{\partial r}(r,w) = \text{as above}$$

$$\frac{\partial^{i+j} f_r}{\partial r^i \partial w_{i_1}\cdots\partial w_{i_j}}(r,w) = \begin{cases} 0 & \text{if } r<\frac{3}{4} \text{ and if } r=1-2^{-2k} \\ \frac{\partial^j h_k}{\partial w_{i_1}\cdots\partial w_{i_j}}(w) \cdot \frac{d^{i-1} g_k}{dr^{i-1}}(r) & \text{if } 1-2^{-2k}<r<1-2^{-2(k+1)} \end{cases} \quad (i,j\geq 1 \text{ or } i\geq 2).$$

Because of I and II, the continuity of all these partial derivatives follows as in 1. Their differentiability with respect to any w_i is clear; as to their differentiability with respect to r, because of III and IV, this follows as in 3. Thus by induction, f is C^∞.

7. <u>f^{-1} is C^∞.</u>

This is the case if
$$J = \begin{vmatrix} \frac{\partial f_r}{\partial r} & \frac{\partial f_r}{\partial w_1} & \cdots & \frac{\partial f_r}{\partial w_{n-1}} \\ \frac{\partial f_{w_1}}{\partial r} & \frac{\partial f_{w_1}}{\partial w_1} & \cdots & \frac{\partial f_{w_1}}{\partial w_{n-1}} \\ \vdots & & & \vdots \\ \frac{\partial f_{w_{n-1}}}{\partial r} & \frac{\partial f_{w_{n-1}}}{\partial w_1} & \cdots & \frac{\partial f_{w_{n-1}}}{\partial w_{n-1}} \end{vmatrix} \neq 0 \text{ on } E\ \{0\}.$$

[For r=0 the statement is again trivial].

But indeed
$$J = \begin{vmatrix} \frac{\partial f_r}{\partial r} & & & & & \frac{\partial f_r}{\partial w_{n-1}} \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{vmatrix} = \frac{\partial f_r}{\partial r} > 0$$
as is proved in 3.

THEOREM 3 (Dieudonné): Every paracompact space is normal.

THEOREM 4 (Bourbaki): Let $\mathcal{U}=\{U_i\}_{i\in I}$ be a locally finite open covering of a normal Hausdorffspace X, then an open covering $\mathcal{W}=\{W_i\}_{i\in I}$ (same indexset) of X exists, with $\overline{W}_i \subset U_i$ for all $i\in I$.

PROOFS: See, for instance, C. Teleman: Grundzüge der Topologie und differenzieerbare Mannigfaltigkeiten, pages 115 and 117. [Berlin, VEB Deutscher Verlag der Wissenschaften, 1968].

THEOREM 5: Every open covering of a paracompact differentiable manifold has a good refinement.

PROOF: Let $\mathcal{U}''$ be an open covering of a paracompact differentiable manifold X, $\mathcal{U}'$ a refinement of $\mathcal{U}''$, so that every $U'\in\mathcal{U}'$ is contained in *one* coördinate neighbourhood, and $\mathcal{U}$ a locally finite refinement of $\mathcal{U}'$. Say $\mathcal{U}=\{U_i\}_{i\in I}$; for each $i\in I$ there exists a diffeomorphism $\psi_i: U_i \to E_i \subset R^n$. According to theorems 3 and 4, an open covering $\mathcal{W}=\{W_i\}_{i\in I}$ of X exists with $\overline{W}_i \subset U_i$ for all $i\in I$.

Since $\mathcal{U}$ is locally finite, there exists around each point $x\in X$ an open neighbourhood $V_x''\ni x$ which has points in common with only finitely many sets U_i. Now for each point $x\in X$ define $V_x' = V_x''\cap(\bigcap_{U_i\ni x} U_i)\cap(\bigcap_{W_i\ni x} W_i)\cap(\bigcap_{U_i\not\ni x} \overline{W}_i^{\text{complement}})$.

Because $\overline{W}_i^c\supset U_i^c$ this intersection does not change if all those $\overline{W}_i^c$'s are left out for which $U_i\cap V_x'' =\emptyset$ (or: $V_x''\subset U_i^c$), so V_x' can be regarded as a finite intersection of open sets containing x, and therefore is an open set containing x itself.

Write $I_x = \{i\in I, U_i\ni x\}$; note that $I_x\neq\emptyset$, and choose some element $i_0\in I_x$. For each other element $i\in I_x$ we can use theorem 1 in the following way. Let $x_0:=\psi_{i_0}(x)$, $E:=\psi_{i_0}(V_x')$, $V_{(i)}:=\psi_i(V_x')$ and $f_{(i)}:E\to V_{(i)}$ be the diffeomorfism $\psi_i\circ\psi_{i_0}^{-1}|_E$. Now a $\delta_i>0$ exists, such that for every $\delta>0$ with $\delta<\delta_i$, $B(\psi_{i_0}(x),\delta)\subset\psi_{i_0}(V_x')$ and $\psi_i\circ\psi_{i_0}^{-1}(B(\psi_{i_0}(x),\delta))$ is convex. Let δ_j be the smallest of all δ_i we have found this way, and set $V_x = \psi_{i_0}^{-1}(B(\psi_{i_0}(x),\delta))$, then $x\in V_x\subset V_x'$ and $\psi_i(V_x)$ is convex for all $i\in I_x$.

It is easily checked that for $V_x\ni x$ the following holds (for each $x\in X$):

1) If $x\in W_i$ then $V_x\subset W_i$

2) If $x\notin U_i$ then $V_x\subset\overline{W}_i^c$

3) If $x\in U_i$ then $V_x\subset U_i$ and $\psi_i(V_x)$ is convex.

Claim: $\{V_x\}_{x\in X}$ is a good refinement of $\mathcal{U}''$.

Proof: For each $x\in X$ there exists a U_i with $x\in U_i$ and (by 3)) $V_x\subset U_i$, hence $\{V_x\}_{x\in X}$ is a refinement of $\mathcal{U}$ and therefore of $\mathcal{U}''$.

Now suppose $V_{x_0}\cap\ldots\cap V_{x_k}\neq\emptyset$ ($k\in N^0$). Choose $i\in I$ such that $x_0\in W_i$. By 1) we have $V_{x_0}\subset W_i$ and there is no x_j ($0<j\leq k$) with $x_j\notin U_i$, since for such a x_j 2) would imply that $V_{x_j}\subset\overline{W}_i^c\subset W_i^c\subset V_{x_0}^c$, contradicting $V_{x_0}\cap\ldots\cap V_{x_j}\cap\ldots\cap V_{x_k}\neq\emptyset$.

Thus for $j=0,\ldots,k$ $x_j\in U_i$ and, by 3), $V_{x_j}\subset U_i$ and $\psi_i(V_{x_j})$ is convex. Therefore also $\psi_i(V_{x_0})\cap\ldots\cap\psi_i(V_{x_k}) = \psi_i(V_{x_0}\cap\ldots\cap V_{x_k})$ is convex, which means that $V_{x_0}\cap\ldots\cap V_{x_k}$ is diffeomorphic with a non-empty open convex subset of R^n, and therefore, by theorem 2, also with R^n itself.