CS 154	**Introduction to Automata and Complexity Theory**	**Handout 5**
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Solution Set 1

1. (a) T
   (b) F
   (c) T
   (d) T
   (e) T
   (f) T

2. Let $L_1 = (a \cup b)*$ and L_2 be the respective language in each part.
   (a) Not Equal. L_1 forms "ab" but L_2 doesn't.
   (b) Not Equal. L_1 forms the empty string but L_2 doesn't.
   (c) Not Equal L_1 forms "a" but L_2 doesn't
   (d) Equal The left half of L_2 (i.e. a*) forms all sequences with no "b"'s. The right half forms all other sequences (that L_1 can form).

3.

$$\begin{aligned} f = O(g) &\iff (\exists c_1)(\exists N_1)(\forall n \geq N_1)[f(n) \leq c_1 * g(n)] \\ g = O(h) &\iff (\exists c_2)(\exists N_2)(\forall n \geq N_2)[g(n) \leq c_2 * h(n)] \end{aligned}$$

combining the two sentences and noting the transitive property of $\leq$, we have:

$$(\exists c_1)(\exists c_2)(\exists N_1)(\exists N_2)(\forall n)[(n \geq N_1 \wedge n \geq N_2) \Rightarrow f(n) \leq c_1 * c_2 * h(n)]$$

thus

$$(\exists c_3)(\exists N_3)(\forall n \geq N_3)[f(n) \leq c_3 * h(n)] \quad \text{where } c_3 = c_1 * c_2 \text{ and } N_3 = \max(N_1, N_2)$$

and hence $f(n) = O(h(n))$

4. (Graded by David Rieber, William Toves, and Johnathan Joseph Hart)

   Let p be an integer greater than or equal to 2. The p-adic representation of a positive integer n is unique.

   Existence: For fixed, but arbitrary $p \geq 2$ we prove that every natural n has a p-adic representation by induction.

   Let $P(n)$ denote that the natural number $n > 0$ has a p-adic representation.

   Base case Prove $P(1)$: Since $p^0 = 1$, the p-adic representation of the natural number 1 is 1.

   Inductive step Assume $P(k)$ and show $P(k+1)$, (where $1 \leq k$) By our inductive hypothesis, which states that $P(k)$ holds, we know that k has a p-adic representation. This representation is:

$$a_m a_{m-1} \ldots a_1 a_0 \qquad m \geq 0$$

   such that:

$$k = \Sigma_{i=0}^{m} a_i p^i \qquad \text{where } 1 \leq a_i \leq p$$

Either for every $i = 0, \ldots, m$, $a_i = p$, or there is a smallest j, such that $a_j < p$, and $a_i = p$ for $i < j$. In the first case:

$$k + 1 = \Sigma_{i=0}^{m+1} 1 \cdot p^i$$

thus, the p-adic representation of $k+1$ is $\underbrace{111\ldots1}_{m+1}$. In the second case:

$$k + 1 = \Sigma_{i=0}^{j-1} p^i + (a_j + 1)p^j + \Sigma_{i=j+1}^{m} a_i p^i$$

thus, the p-adic representation of $k+1$ in this case is $m a_{m-1} \ldots a_{j+1}(a_j + 1)1 \ldots 1$.

Thus $P(k+1)$ holds, and so all natural numbers $n > 0$ have a p-adic representation.

Since our proof assumes nothing specific about our choice of p, we can generalize to say that for all $p \geq 2$, all natural numbers $n > 0$ have a p-adic representation.

Uniqueness: Let $a_r a_{r-1} \ldots a_1 a_0$ and $b_s b_{s-1} \ldots b_1 b_0$ be two representations of the same integer n. We show that they are equal. As they both represent n we have

$$\Sigma_{i=0}^{r} a_i p^i = \Sigma_{j=0}^{s} b_j p^j$$

We first show that $r = s$. To prove this suppose $r > s$. Since $1 \leq a_i, b_j \leq p$ for $i = 0 \ldots r$ and $j = 0 \ldots s$, then

$$\Sigma_{i=0}^{r} a_i p^i \geq \Sigma_{i=0}^{r} p^i = \frac{p^{r+1} - 1}{p - 1},$$

and

$$\Sigma_{j=0}^{s} b_j p^j \leq \Sigma_{j=0}^{s} p^{j+1} = p\frac{p^{s+1} - 1}{p - 1}.$$

Now, since $s < r$:

$$\Sigma_{j=0}^{s} b_j p^j \leq p\frac{p^{s+1} - 1}{p - 1} \leq p\frac{p^{r} - 1}{p - 1} = \frac{p^{r+1} - p}{p - 1} < \frac{p^{r+1} - 1}{p - 1} \leq \Sigma_{i=0}^{r} a_i p^i$$

which contradicts our assumption, thus $r = s$.

Let k be the largest integer such that $a_i \neq b_i$. Then:

$$a_k p^k + \Sigma_{i=0}^{k-1} a_i p^i = b_k p^k + \Sigma_{i=0}^{k-1} b_i p^i$$

and

$$(a_k - b_k)p^k = \Sigma_{i=0}^{k-1} (b_i - a_i) p^i$$

Suppose (without loss of generality) that $a_k > b_k$. Then $a_k - b_k \geq 1$ and

$$(a_k - b_k)p^k \geq p^k$$

On the other hand, $b_i - a_i \leq (p-1)$ for $i = 0 \ldots k-1$. Thus,

$$\Sigma_{i=0}^{k-1} (b_i - a_i) p^i \leq \Sigma_{i=0}^{k-1} (p-1) p^i = p^k - 1$$

but this is a contradiction. Thus, we have proved that for every positive integer n, if it has a p-adic representation, that representation is unique.

5. Solutions to Problems 5 & 6 exist only in hardcopy form. Copies are in the Theory Handout Hangout.

Extra credit problem

1. *Prove that in any group of six students taking this class, there is a group of three people who are all either mutual strangers or all mutual acquaintences. You may assume that all pairs of students in this class are either mutual strangers or mutual acquaintences.*

   Consider any student (call her A). Every one of the other 5 students is either a friend or an acquaintance, by assumption. By the pigeon-hole principle, she should therefore have at least 3 friends or at least 3 acquaintances (otherwise it wouldn't add up to 5). Let us assume, without loss of generality that she has at least 3 friends. Choose some 3 of them, and consider the nature of relationships between them. If there is at least one friend-friend pair between them, we are done, as the group consisting of A and these two form a threesome of mutual friends. If, on the other hand, the friends of A are all mutual acquaintances (no friend-friend pair among them) we are again done.

2. *Prove that in a class of infinitely many students, either there is an infinite subset of mutual strangers or there is an infinite subset of mutual aquaintances.*

   This is a famous theorem known as Ramsey's theorem. One can formulate and verify a version of this theorem within *monadic second order logic of one successor* (S1S) which is a decidable logic. The decision procedure for S1S is by the way obtained using regular automaton on infinite words (introduced in problem 4.10-10). Thus, one can make a computer solve this problem "easily".

   An informal proof of the theorem can take the following lines:

   First color all the students pink and pick one pink student. By the pigeon hole principle this student has either infintely many acquaintances (A) or infinitely many strangers (S) among the other students. If S is infinite, then color the students in A green, put the chosen student in a bucket labeled σ. On the other hand if S is finite, then A is infinite, color the S-students green, and put the chosen student in a bucket labeled α. Continue this process transfinitely with the remaining infinite set of pink students at each step picking a pink student and partitioning the rest of according to whether they are in A or in S relative to the chosen student. In the end either the bucket labeled α or the bucket labeled σ will contain infinitely many students. The students in α will all be mutual aquaintances, and the students in σ will all be mutual strangers by the construction. □